THE INVOLUTION CURVE DETERMINED FROM A SPECIAL PENCIL OF n-ICS*

BY

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Introduction. In *The geometry of the Weddle manifold* W_p [1]† there arises the necessity for investigating involution curves, some of which cover W_p, while others cover the sections of W_p by its F-loci. Since the points of an involution curve V_k are in (1, 1) correspondence with the k-ads from the members of a pencil of n-ics, V_k may be constructed as follows: Assign the parameter t to the points of a norm curve N^k in S_k. The hyperplanes of the curve given by a general member of the pencil intersect in $C_{n,k}$ points, the locus of which is the curve V_k of order $C_{n-1,k-1}$. The involution curves of W_p are particularized by the fact that the pencil of n-ics contains one member with a number of double roots. In the study of these it is important to determine the genus; it is likewise necessary to discover what multiple points are introduced by the special n-ic if we are to ascertain the effect of the Cremona transformations associated with W_p. The article just cited makes no attempt to solve these problems except for special values of n, k. In the present paper (§§1–4) we solve them for general n, k where one n-ic of the pencil possesses j distinct double roots $(1 \leqq j \leqq n/2)$.

We determine the genus of V_k by applying to the correspondence set up between the points of N^k and the points of V_k Zeuthen's formula:

$$\eta - \eta' = 2\alpha(p'-1) - 2\alpha'(p-1).$$

Here α, α' are the indices of the correspondence, p, p' the genera of the curves, and η, η' the numbers of branch points of the curves. η, η' may be calculated in either of two ways. By the first method, the point t_1 of N^k is counted $v_1 - 1$ times as a branch point if v_1 of the points y of V_k corresponding to t_1 coalesce at γ on the same branch of V_k. Hence we must first investigate the multiple points of V_k. This method I use in §1, and used originally in §2. The second much more elegant method, which may be applied at once without a study of the multiple points, was pointed out to me by O. Zariski. It depends on the following criterion for the multiplicity of a coincidence in an (α, α') correspondence between two algebraic curves V and V': Let P_0 be a point on the

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† Numbers in brackets refer to the bibliography at the end of the paper.

Riemann surface of V; as a variable point P of V, very near P_0, turns around P_0, the corresponding α' points on V' may be permuted among themselves. If this permutation consists of cycles of periods $v_1, v_2, \cdots, v_m$, $(\sum_{i=1}^{m} v_i = \alpha')$, then to the point P_0 there corresponds on V', m coincidences of multiplicities $v_1, v_2, \cdots, v_m$ respectively. P_0 then contributes to η the value $\sum_{i=1}^{m}(v_i - 1)$. By means of this method we readily study the special cases of §6, where some or all of the j double roots of the special n-ics coincide. It is interesting to note that η, η' are not always identical for the two definitions, though their difference is the same for each.

In §3 we determine the multiple points of V_k for $j>1$ by comparing the value of its genus as obtained in §2 to the value determined by the first method. In §4 we check this result by a projective study of V_k as part of the intersection Γ of $k-1$ hypersurfaces. These hypersurfaces are represented by the vanishing of $k-1$ determinants of highest order selected in a special way from a matrix whose elements are binary n-ics. The order of the manifold represented by the vanishing of all the determinants of highest order in such a matrix has been studied by Brill [2] as the order of a restricted system of equations. One reason for the introduction of this second proof is the light that it throws upon such manifolds; in particular the machinery set up for our work shows that Brill's formulas may be more easily obtained from the study of a dual matrix whose elements are linear forms in several variables. We thus prove that Brill's problem reduces to a special case of those investigated by Salmon, Roberts, Cayley, and Pieri. A more important reason for the projective method is that it introduces interesting manifolds associated with V_k; also it seems adapted to the study of manifolds which are obtained from V_k by generalization.

In §5 we prove an interesting identity involving binomial coefficients which I have not found in the literature.

1. **The involution curve of the general pencil.** The pencil of n-ics $[(a_1t)^n, (a_2t)^n]$ contains $2(n-1)$ members with a double root each. We have for N^k, V_k:

$$(1) \qquad \alpha = k, \qquad \alpha' = C_{n-1,k-1}, \qquad p = 0, \qquad p' = p_k.$$

Since the involution curve has no multiple points, we easily calculate η, η' by the first method:

$$(2) \qquad \eta = 2(n-1)(k-1)C_{n-2,k-1}; \qquad \eta' = 2(n-1)C_{n-2,k-2}.$$

Substituting these values in Zeuthen's formula, we have

$$(3) \qquad p_k = (n-k)C_{n-1,k-1} - C_{n,k} + 1.$$

This is unaltered when k and $n-k$ are interchanged, checking the fact that k-ads and complementary $(n-k)$-ads in the pencil of n-ics determine two curves in (1, 1) correspondence.

2. **The genus of the involution curve V_k for which one n-ic possesses several double roots.** Let the pencil contain one n-ic with j distinct double roots, while there are $2(n-1)-j$ n-ics with a double root each. Since the n-ics of the latter type behave as before with respect to η, η', we have for these

$$\eta_0 = (2n - 2 - j)(k - 1)C_{n-2,k-1}, \qquad \eta_0' = (2n - 2 - j)C_{n-2,k-2}$$

so that

$$\eta_0 - \eta_0' = (2n - 2 - j)(n - k - 1)C_{n-2,k-2}. \tag{1}$$

To determine the special n-ic's contributions η_s, η_s' to η, η', we use Zariski's argument.

Let the special n-ic of the pencil correspond to the value x_0 of the parameter x. As x very near x_0 turns around x_0, the roots of the n-ic corresponding to x will be permuted, the permutation consisting of j cycles of period two and $n-2j$ cycles of period one. For convenience we write this permutation

$$S = (t_{11}t_{12}) \cdots (t_{i1}t_{i2}) \cdots (t_{j1}t_{j2})(\tau_1)(\tau_2) \cdots (\tau_{n-2j}). \tag{2}$$

A corresponding permutation Σ takes place among the k-ads of t's, τ's chosen from the roots of the n-ic. As x approaches x_0, a letter of the ith cycle approaches the point P_i of N^k corresponding to the double root of the special n-ic, and τ_h approaches the point Q_h of N^k. The k-ads of t's and τ's approach points $\pi(t, \tau)$ of V_k. To find the number of coincidences on V_k corresponding to P_i or Q_h, we let t_i or τ_h turn around P_i or Q_h, and write down the corresponding permutation on the k-ads involving t_i or τ_h. From this it is seen that the P_i are not branch points, since the corresponding permutations are induced by Σ^2, and Σ is of period two. The permutations corresponding to Q_h are induced by Σ, under which the k-ads actually permuted are those and only those which contain one only of some pair $(t_{i1}t_{i2})$. Such a k-ad is interchanged with another as τ_h turns about Q_h. The total number of k-ads containing one and only one point from each of r pairs (t_{i1}, t_{i2}), $1 \leqq r \leqq j$, and $k-r-2l$ points τ_h, $0 \leqq l \leqq j-r$, is $C_{j,r}\, C_{j-r,l}\, C_{n-2j,k-r-2l}2^r$. Since each such k-ad contains $k-r-2l$ branch points, and since the k-ads are permuted in pairs by Σ, we have

$$\eta_s = \sum_{r=1}^{j} \sum_{l=0}^{j-r} C_{j,r}C_{j-r,l}C_{n-2j,k-r-2l}(k - r - 2l)2^{r-1}. \tag{3}$$

A point $\pi(t, \tau)$ of V_k, for which the k-ad turning about it contains one and only one point of a pair $(t_{i1}t_{i2})$ cannot be a branch point, since the corresponding permutation among the t's is induced by S^2. The k-ads containing both members of l such pairs and $k-2l$ points $\tau_h(1\leqq l\leqq i)$ are $C_{j,l}\,C_{n-2j,k-2l}$ in number. Hence

$$\eta_s' = \sum_{l=1}^{j} C_{j,l}C_{n-2j,k-2l}l. \tag{4}$$

The decrease in p_k due to the presence of the special n-ic is therefore

$$\begin{aligned}\tilde{p} = \frac{1}{2k}\Big\{ j(n-k-1)C_{n-2,k-2} &+ \sum_{l=1}^{j} C_{j,l}C_{n-2j,k-2l}l \\ &- \sum_{r=1}^{j}\sum_{l=0}^{j-r} C_{j,r}C_{j-r,l}C_{n-2j,k-r-2l}(k-r-2l)2^{r-1}\Big\},\end{aligned} \tag{5}$$

and

$$p_k = (n-k)C_{n-1,k-1} - C_{n,k} + 1 - \tilde{p}. \tag{6}$$

3. **The multiple points of V_k.** The special n-ic introduces whatever multiple points exist. We determine what their nature must be to produce the value $\eta_s-\eta_s'$ obtained from §2, (3), (4). Let the special n-ic be $(ts_1)(ts_1)\cdots(ts_j)(ts_j)(tt_1)(tt_2)\cdots(tt_{n-2j})$. Then the point $P_l(s_1, s_1, s_2, s_2, \cdots, s_l, s_l, t_1, t_2, \cdots, t_{k-2l})$ is a simple point on V_k and contributes 0 to η_s and l to η_s'. There are $\sum_{l=1}^{j}C_{j,l}\,C_{n-2j,k-2l}$ such points. At the point

$$P_{rl}(s_1, s_2, \cdots, s_r, s_{r+1}, s_{r+1}, \cdots, s_{r+l}, s_{r+l}, t_1, t_2, \cdots, t_{k-r-2l})$$

there are 2^{r-1} or fewer branches, since $s_i(i\leqq r)$ meets V_k here in at most 2^{r-1} points. If there are 2^{r-1} branches, the contribution to η_s due to the t's and the s_i's $(i>r)$ at the point is $(k-r-l)\;2^{r-1}$; the s_i's $(i\leqq r)$ contribute nothing. On the other hand, the contribution to η_s' due to P_{rl} is $l\cdot 2^{r-1}$. If the number of branches is smaller, the contributions to η_s due to this point increase, while that to η_s' decreases. There are $\sum_{r=1}^{j}\sum_{l=0}^{j-r}C_{j,r}\;C_{j-r,l}\;C_{n-2j,k-r-2l}$ such points. Hence only if there are 2^{r-1} branches at each point P_{rl}, can we have

$$\eta_s - \eta_s' = \sum_{r=1}^{j}\sum_{l=0}^{j-r} C_{j,r}C_{j-r,l}C_{n-2j,k-r-2l}(k-r-2l)2^{r-1} - \sum_{l=1}^{j} C_{j,l}C_{n-2j,k-2l}l, \tag{1}$$

the value obtained from §2, (3), (4).

THEOREM 1. *The point P_{rl} described above is a 2^{r-1}-fold point on V_k; in the calculation of the genus it is equivalent to $(r-1)\;2^{r-2}$ ordinary double points.*

The first statement we have just proved. If the second holds, the decrease in the genus due to the special n-ic must be written:

$$\tilde{p} = \sum_{r=2}^{j} \sum_{l=0}^{j-r} C_{j,r} C_{j-r,l} C_{n-2j,k-r-2l} (r-1) 2^{r-2}. \tag{2}$$

Equating the right-hand members of (2) and §2, (5) leads to the following, after simplification:

$$j(n-k-1) C_{n-2,k-2} = \sum_{r=0}^{j} \sum_{l=0}^{j-r} C_{j,r} C_{j-r,l} C_{n-2j,k-r-2l} (kr - r - 2l) 2^{r-1}. \tag{3}$$

That (3) is satisfied under the conditions of our problem is proved in §5.

4. The study of V_k from a projective view point; its multiple points and the manifolds associated with it. Let the pencil of n-ics $[(a_1t)^n, (a_2t)^n]$ be contained in a linear system $\Sigma_{k-1} = [(a_1t)^n, (a_2t)^n, \cdots, (a_kt)^n]$. The k-ads of the pencil represent the points of a curve on the surface M^{n-k+1} of order $n-k+1$ given by the equation

$$\begin{vmatrix} (a_1t_1)^n & (a_2t_1)^n \cdots & (a_kt_1)^n \\ (a_1t_2)^n & (a_2t_2)^n \cdots & (a_kt_2)^n \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ (a_1t_k)^n & (a_2t_k)^n \cdots & (a_kt_k)^n \end{vmatrix} \cdot \frac{1}{\pi(t_it_j)} = 0, \tag{1}$$

where $t_1, t_2, \cdots, t_k$ are interpreted as Darboux coordinates in the space S_k.

Take the matrix

$$L_1 = \begin{Vmatrix} (a_1t_1)^n & (a_2t_1)^n \cdots & (a_{2k-2}t_1)^n \\ (a_1t_2)^n & (a_2t_2)^n \cdots & (a_{2k-2}t_2)^n \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ (a_1t_k)^n & (a_2t_k)^n \cdots & (a_{2k-2}t_k)^n \end{Vmatrix},$$

where the n-ics are linearly independent. Hence $k \leqq (n+3)/2$; if not, we study instead the curve determined by the complementary $(n-k)$-ads. L_1 may be written briefly as $(a_1a_2 \cdots a_{2k-2})$. From L_1 select any set of $k-1$ determinants such that the only columns common to all are (a_1a_2), while each new determinant is obtained from that last found by dropping the third column and inserting as final column one not previously used. A typical set is $(a_1a_2 \cdots a_k)$, $(a_1a_2a_4 \cdots a_{k+1})$, $\cdots$, $(a_1a_2a_{k+1} \cdots a_{2k-2})$. Such a set we define as a set S of determinants.

The linearly independent n-ics $(a_1t)^n, \cdots, (a_mt)^n$, $m \leqq n+1$, determine $n+1$ linearly independent n-ics, $(c_1t)^n, (c_2t)^n, \cdots, (c_{n+1}t)^n$, such that $(c_it)^n$ is apolar to $(a_jt)^n$, $i \neq j$. In S_n the hyperplane $(a_jt)^n$ is on the point $(c_it)^n$, $i \neq j$. Thus the k-ad $(t_1, t_2, \cdots, t_k)$ making vanish the determinants of highest order of the matrix $(a_1a_2 \cdots a_m)$, $m \leqq k$, represents an S_{k-1} (on the points $t_1, \cdots, t_k$

of the norm curve N^k) joined to the $S_{n-m}[(c_{m+1}t)^n, (c_{m+2}t)^n, \cdots, (c_{n+1}t)^n]$ by an S_{n-1}. Hence the S_{k-1} meets the S_{n-m} in an S_{k-m}; that is, the k-ic $(qt)^k = (tt_1)(tt_2) \cdots (tt_k)$ is apolar to $k-m+1$ independent members of the linear system $[(c_{m+1}t)^n, \cdots, (c_{n+1}t)^n]$. If $\sum_{i=m+1}^{n+1}\lambda_i(c_iq)^k(c_it)^{n-k} \equiv 0$, we have $n-k+1$ equations in $n-m+1$ λ's. If there are $k-m+1$ independent solutions, the rank of the matrix of the system must be $n-k$. Thus the determinants of highest order in the matrix must vanish, which we indicate by writing $[c_{m+1}c_{m+2} \cdots c_{n+1}] \equiv 0$. The k-ad $(t_1, t_2, \cdots, t_k)$ making vanish the determinants of highest order of a matrix $(a_1a_2 \cdots a_m)$, $m > k$, represents an S_{k-1} such that there are only $k-1$ conditions imposed for any hyperplane which is a linear combination of $(a_1t)^n, (a_2t)^n, \cdots, (a_mt)^n$ to be on the S_{k-1}. The S_{k-1} must therefore meet the $S_{n-m}[(c_{m+1}t)^n, (c_{m+2}t)^n, \cdots, (c_{n+1}t)^n]$ in a point; that is, $(qt)^k = (tt_1)(tt_2) \cdots (tt_k)$ is apolar to a member of the linear system $[(c_{m+1}t)^n, (c_{m+2}t)^n, \cdots, (c_{n+1}t)^n]$. As before, we have $n-k+1$ equations in $n-m+1$ λ's. For a solution to exist, the determinant must vanish for each set of $n-m+1$ equations. Thus again the matrix $[c_{m+1}c_{m+2} \cdots c_{n+1}] \equiv 0$. We note here that in either case the manifold represented by the matrix $(a_1a_2 \cdots a_m) \equiv 0$, where the elements are binary n-ics, may likewise be represented by the matrix $[c_{m+1}c_{m+2} \cdots c_{n+1}] \equiv 0$, where the elements are linear in the coefficients of $(qt)^k$. Hence the formulas developed by Brill for the order of a manifold $(a_1a_2 \cdots a_m) \equiv 0$ (cf. [2], pp. 391–395), may be more easily determined from the dual matrix $[c_{m+1}c_{m+2} \cdots c_{n+1}] \equiv 0$. For if we interpret the coefficients of $(qt)^k$ as variables, the latter becomes a matrix whose elements are linear forms. Such a matrix of i rows and j columns $(i \geqq j)$ is of order $C_{i,j-1}$ (cf. [3], p. 63).

We state without proof two preliminary theorems, the first of which is well known.

THEOREM 1. *If in a matrix of m rows and n columns* $(m<n)$ *every determinant of the mth order containing a certain set of i columns vanishes, while the i columns are not linearly dependent, then every determinant of the mth order in the matrix vanishes.*

THEOREM 2. *The determinants of a set S chosen as above from* L_1, *when divided by appropriate products and equated to zero, represent a set of hypersurfaces* $M_{(s)}^{n-k+1}$ *which intersect in a manifold* Γ_s *of dimension one.*

The truth of this theorem is implicitly assumed by Brill [2], though he does not use the geometric interpretation of this article. It may be checked by showing that the t's of a k-ad which represents a point on Γ_s are the roots of a k-ic apolar to a point n-ic in each of the S_{n-k}'s represented by the determinants of S; the $k-1$ conditions imposed on the coefficients of the k-ic can be proved linearly independent. One part of Γ_s is the curve V_k. Another is the curve V_Δ whose points are represented by the k-ads making vanish all the determinants of highest order in L_1.

We now proceed to the fundamental theorems of this section.

THEOREM 3. *No part of V_k (except for a finite number of points) can appear multiply in any Γ_s.*

If a point is of multiplicity two or more for the intersection curve of $k-1$ surfaces in S_k, while it is an ordinary point for each surface, the tangent hyperplanes of the surfaces at that point must have at least an S_2 in common (cf. [4], p. 191); that is, at most only $k-2$ tangent hyperplanes can be linearly independent. It may then be readily proved that at least one surface in the linear system determined by the $k-1$ $M_{(s)}^{n-k+1}$ must have a double point here. However, no surface which is a linear combination of the $M_{(s)}^{n-k+1}$ can have a multiple point at any point determined from a non-singular n-ic of the pencil.

THEOREM 4. *There is no point on V_k common to the curves residual to V_k in all possible Γ_s's determined from every matrix L having the first two columns in common with L_1.*

We show that given V_k determined by the pencil $[(a_1t)^n, (a_2t)^n]$, and $(qt)^k=(tt_1)(tt_2)\cdots(tt_k)$, any k-ic factor of $(a_1t)^n$, we can find a matrix L such that the point $T(t_1, t_2, \cdots, t_k)$ is on Γ_s for every set $M_{(s)}^{n-k+1}$, but is on no curve residual to V_k.

If $k=2$, Γ_s and V_k are identical, so that the theorem is obvious. If $k=3$, L is of the form $(a_1a_2a_3a_4)$. Γ_s is given by the simultaneous vanishing either of $(a_1a_2a_3)$, $(a_1a_2a_4)$, or of the dual c-matrices, $[c_4c_5\cdots c_{n+1}]$, $[c_3c_5\cdots c_{n+1}]$. From the latter representation its order is found to be $(n-2)^2$. V_k is represented by $(a_1a_2)\equiv 0$, or $[c_3c_4\cdots c_{n+1}]\equiv 0$; V_Δ by $(a_1a_2a_3a_4)\equiv 0$, or $[c_5c_6\cdots c_{n+1}]\equiv 0$. Their orders are respectively $C_{n-1,n-3}$, $C_{n-2,n-4}$. Since the order of Γ_s is precisely the sum of those of V_k, V_Δ, our theorem is proved for this case if we can show that a matrix can be found for which T is not on V_Δ. For $k>3$ we determine the conditions to be satisfied if T is on a curve residual to V_k and V_Δ in Γ_s. Γ_s is given by the vanishing of $k-1$ a-matrices, or the $k-1$ dual c-matrices:

$$(2\text{a})\qquad (a_1a_2a_3\cdots a_k),\ (a_1a_2a_4\cdots a_{k+1}),\ \cdots,\ (a_1a_2a_{k+1}\cdots a_{2k-2})\equiv 0,$$

$$(2\text{b})\qquad [c_{k+1}\cdots c_{n+1}],\ [c_{k+2}\cdots c_{n+1}c_3],\ \cdots,\ [c_{2k-1}\cdots c_{n+1}c_3\cdots c_k]\equiv 0.$$

It is made up of V_k, given by $[c_3c_4\cdots c_{n+1}]\equiv 0$, and of irreducible curves each of which is on at least one of the manifolds:

$$(3)\qquad [c_{k+2}\cdots c_{n+1}],\ [c_{k+3}\cdots c_{n+1}c_3],\ \cdots,\ [c_{2k-1}\cdots c_{n+1}c_3\cdots c_{k-1}]\equiv 0.$$

For consider an irreducible curve V' which is part of Γ_s, and for which $[c_{k+2} \cdots c_{n+1}] \not\equiv 0$. By Theorem 1, $[c_{k+1} \cdots c_{n+1}c_3] \equiv 0$ for V'. If none of (3) is satisfied, repeated applications of Theorem 1 prove that for the points of V', $[c_{k+1} \cdots c_{n+1}c_3 \cdots c_k] \equiv 0$; thus V' coincides with all or part of V_k. Since no part of V_k can figure multiply in Γ_s [cf. Theorem 3], a point T on a curve residual to V_k must therefore make vanish one of the matrices of (3), say $[c_{k+2} \cdots c_{n+1}]$. Let us further suppose that T is not on V_Δ, so that $[c_{2k-1} \cdots c_{n+1}] \not\equiv 0$. Then the polar with respect to $(qt)^k$ of a linear combination of $(c_{k+2}t)^n, \cdots, (c_{2k-2}t)^n$ is either identically zero or is the polar of a linear combination of $(c_{2k-1}t)^n, \cdots, (c_{n+1}t)^n$. This means that in S_n the S_{n-k+2} $[(tt_1)^n, \cdots, (tt_k)^n, (c_{2k-1}t)^n, \cdots, (c_{n+1}t)^n]$ meets the S_{k-4} $[(c_{k+2}t)^n, \cdots, (c_{2k-2}t)^n]$ in a point.

We now build a matrix L of the desired type. The hyperplane $(a_1t)^n$ intersects the norm-curve N^k in $t_1, \cdots, t_k$, and is intersected by $(a_2t)^n$ in an S_{n-2}. In this S_{n-2} we choose $n-2k+3$ points $(c_{2k-1}t)^n, \cdots, (c_{n+1}t)^n$, independent of each other and of $(tt_1)^n, \cdots, (tt_k)^n$. This insures that T is not on V_Δ, and if $k=3$, we may choose for $(c_3t)^n$, $(c_4t)^n$ any two further independent points in the S_{n-2}. If $k>3$, we proceed as follows: The points $(tt_1)^n, \cdots, (tt_k)^n$, $(c_{2k-1}t)^n, \cdots, (c_{n+1}t)^n$, determine an S_{n-k+2}. In the S_{n-2} of $(a_1t)^n$ we take an S_{2k-5} meeting the S_{n-k+2} in an S_{k-2}. In the S_{2k-5} but outside the S_{k-2} we choose $2k-4$ independent points $(c_3t)^n, \cdots, (c_{2k-2}t)^n$, such that the S_{k-4} determined by any $k-3$ will not intersect the S_{k-2}. Then T cannot be on any curve residual to V_k in a Γ_s. We take for $(c_1t)^n$, $(c_2t)^n$ any two further independent n-ics, and from these c's the set of a's (including a_1, a_2) may be determined.

THEOREM 5. *If a point is of multiplicity m for all Γ_s's determined from every matrix L, it is of multiplicity m for V_k.*

THEOREM 6. *If an n-ic of the pencil contains k pairs of double roots $t_1=t_2=s_1$, $t_3=t_4=s_2, \cdots, t_{2k-1}=t_{2k}=s_k$, then the point $(s_1, s_2, \cdots, s_k)$ is a double point for the surface M^{n-k+1}.*

Taking the particular n-ic for $(a_1t)^n$ and recalling the equation (1) for M^{n-k+1}, we see that the k lines of type $t_1=s_1$, $t_2=s_2, \cdots, t_{k-1}=s_{k-1}$ meet the surface here in two coincident points. If this is not a double point of the surface, the k lines must lie in a common tangent hyperplane. A point of such a line is $(ts_1)(ts_2) \cdots (ts_{k-1})(tt')$, where t' may vary. The k-ic giving the intersections of the common tangent hyperplane with the normcurve must be apolar to this k-ic and so may be written

$$(\eta t)^k \equiv \sum_{i=1}^{k-1} \lambda_i(ts_i)^k + \lambda_k(tt')^k.$$

But if $(\eta t)^k$ may also be written

$$(\eta t)^k \equiv \sum_{i=1}^{k-1} \mu_i (ts_i)^k + \mu_k (tt'')^k,$$

we must have $\lambda_k = \mu_k = 0$, since there is no identity connecting kth powers of $k+1$ linear forms. Thus the hyperplane is a linear combination of the hyperplanes $(ts_1)^k, (ts_2)^k, \cdots, (ts_{k-1})^k$. Similarly it is a linear combination of any $k-1$ of the hyperplanes $(ts_1)^k, \cdots, (ts_k)^k$. Thus

$$\sum_{i=1}^{k-1} \lambda_i (ts_i)^k \equiv \sum_{j=2}^{k} \mu_j (ts_j)^k,$$

and $\lambda_1 = \mu_k = 0$. Likewise all the other λ's are zero. Hence no such hyperplane exists, and $(s_1, s_2, \cdots, s_k)$ is a double point of the surface.

THEOREM 7. *An S_{k-1} of the normcurve N^k meets the M^{n-k+1} in an $\overline{M}^{n-k+1}$ of similar type determined by a linear system Σ_{k-2} of $(n-1)$-ics related to the normcurve N^{k-1} in S_{k-1}.*

Let the hyperplane S_{k-1} have the parameter t_1 on N^k. It meets M^{n-k+1} in a locus of points with fixed coordinate t_1. The $(k-1)$-ads giving the other coordinates are obtained after factoring (tt_1) from a system $(\lambda_1, \cdots, \lambda_k)$, where $\sum_{i=1}^{k} \lambda_i (a_i t_1)^n = 0$. Due to this relation we have a linear system Σ_{k-2} determined by $k-1$ $(n-1)$-ics. Using also the theorem that the S_{k-1}'s of a normcurve N^k cut out on a fixed hyperplane of N^k the S_{k-2}'s of an N^{k-1} in the hyperplane, we may set up for $\overline{M}^{n-k+1}$ an equation similar to (1). By continuing this process, we arrive at the general theorem:

THEOREM 8. *The S_{k-r} $(r=1, 2, \cdots, k-2)$ which is the intersection of r hyperplanes $t_1, t_2, \cdots, t_r$ of N^k meets the M^{n-k+1} in an $\overline{M}^{n-k+1}$ of similar type determined by a system Σ_{k-r-1} of $(n-r)$-ics related to the normcurve N^{k-r} in the S_{k-r}.*

THEOREM 9. *If the pencil contains an n-ic $(ts_1)(ts_1)(ts_2)(ts_2) \cdots (ts_j)(ts_j)$ $(tt_1)(tt_2) \cdots (tt_{n-2j})$, then at any point P_{rl} $(s_1, s_2, \cdots, s_r, s_{r+1}, s_{r+1}, \cdots, s_{r+l}, s_{r+l}, t_1, t_2, \cdots, t_{k-r-2l})$, the space S_r determined by the intersection of hyperplanes $s_{r+1}, s_{r+1}, \cdots, s_{r+l}, s_{r+l}, t_1, t_2, \cdots, t_{k-r-2l}$ is tangent to M^{n-k+1} (the intersection of s_i with s_i is the S_{k-2} of N^k at s_i).*

For by Theorems 7 and 8 this S_r meets M^{n-k+1} in an $\overline{M}^{n-k+1}$ with a double point at P_{rl}. Hence any line in S_r through P_{rl} is tangent to M^{n-k+1} at that point.

THEOREM 10. *The point P_{rl} of Theorem 9 is a point of multiplicity 2^{r-1} at least for all Γ_s's determined from every matrix L.*

If $r=k$, the point is a double point for all M^{n-k+1}'s [cf. Theorem 6], and is therefore of multiplicity 2^{k-1} at least for the curve determined by any $k-1$ of the M^{n-k+1}'s. If $r<k$, P_{rl} must be a simple point for the general M^{n-k+1}. However, the tangent hyperplanes to the M^{n-k+1}'s at this point have a common S_r. Hence we may substitute for the original M^{n-k+1}'s of a set $M_{(s)}^{n-k+1}$ a system of $k-1$ independent M^{n-k+1}'s in which $r-1$ have double points at P_{rl}. Their intersection has therefore at P_{rl} a point of multiplicity 2^{r-1} at least ([4], p. 191).

THEOREM 11. *The point P_{rl} above described is a point of multiplicity exactly 2^{r-1} for V_k.*

That the multiplicity is at least 2^{r-1} is evident from Theorems 5 and 10. That V_k has no higher multiplicity at P_{rl} is shown by the fact that any hyperplane s_i $(i\leqq r)$ meets it here in only 2^{r-1} points.

We finally want to prove

THEOREM 12. *The restriction $k\leqq(n+3)/2$ imposed on the curve V_k in this section is valid for the generic curves $[W_p, S_p]$ of W_p.*

The curves meant are those cut out on W_p by $(p+1)$-secant S_p's of N^{2p-1} as defined in [1], §5. They are related to a pencil of $(2p-j)$-ics containing one member with $p-j-1$ double roots $(-2\leqq j\leqq p-1)$. For $j\geqq 0$, they are involution curves determined by $(p-j)$-ics of the pencil (cf. [1], §§4, 6, (4), (7)); for $j=-2, -1$, they are projections of such curves (cf. [1], §6, (9)). Since $p-j\leqq(2p-j+3)/2$ for $j\geqq -3$, the restriction is satisfied. Furthermore, for $j\leqq 3$, the restriction is also satisfied for the involution curve determined from the p-ics of the pencil, which is a Cremona transform of the generic curve (cf. [1], §6, (11)).

5. **Proof of the identity** (3) **in** §3. The identity is obviously true for $j=0$. In the following proof we assume $n\geqq 2j, j\geqq 1, n\geqq k\geqq 1$. Wherever they occur, we take $C_{0,0}=1$ and $C_{a,b}=0$, if $a<b$. These conditions insure that the terms of the right-hand member involve true binomial coefficients and are restrictions obviously imposed by the geometric aspects of our problem. We have

$$
\begin{aligned}
&\sum_{r=0}^{j}\sum_{l=0}^{j-r} C_{j,r}C_{j-r,l}C_{n-2j,k-r-2l}(kr-r-2l)2^{r-1} \\
(1)\qquad &= j\sum_{r=0}^{j-1}2^r\sum_{l=0}^{j-r-1}\frac{(j-1)!}{r!l!(j-r-l-1)!}\left[C_{n-2j,k-r-2l-1}(k-1)-C_{n-2j,k-r-2l-2}\right].
\end{aligned}
$$

If in

$$
\sum_{r=0}^{j-1}2^r\sum_{l=0}^{j-r-1}\frac{(j-1)!}{r!l!(j-r-l-1)!}C_{n-2j,k-r-2l-1}
$$

we put $j-1=j'$, $n-2=n'$, $k-1=k'$, we get

$$\sum_{r=0}^{j'} 2^r \sum_{l=0}^{j'-r} \frac{j'!}{r!l!(j'-r-l)!} C_{n'-2j',k'-r-2l}. \tag{2}$$

Expanding and rearranging (2) in the form $\sum_{r=0}^{2j'} c_r C_{n'-2j',k'-r}$, we have

$$c_r = \sum_i 2^{r-2i} C_{j',r-i} C_{r-i,i}, \text{ for } r \leqq j', \tag{3a}$$

where $i=0, 1, 2, \cdots, (r-1)/2$ or $r/2$, according as r is odd or even;

$$c_r = \sum_i 2^{2j'-r-2i} C_{j',2j'-r-i} C_{2j'-r-i,i}, \text{ for } r > j', \tag{3b}$$

where $i=0, 1, 2, \cdots, (2j'-r-1)/2$ or $(2j'-r)/2$, according as r is odd or even. Since $2j'-r<j'$ if $r>j'$, both forms are of similar type. Using the identity $C_{j',r-i}C_{r-i,i}=C_{j',i}C_{j'-i,r-2i}$ (cf. [5], 2⁰, p. 178), we transform (3a) into

$$c_r = \sum_i 2^{r-2i} C_{j',i} C_{j'-i,r-2i}, \quad i = 0, 1, 2, \cdots, (r-1)/2, \text{ or } r/2. \tag{3a$'$}$$

We now prove that (3a)$'$ gives us the binomial coefficient $C_{2j',r}$. Let $2j'$ objects be divided into j' pairs. To a set of r taken, let i pairs $(i \leqq r/2)$ contribute both their members, while $r-2i$ pairs contribute one member each. The i pairs may be chosen in $C_{j',i}$ ways, the $r-2i$ pairs in $C_{j'-i,r-2i}$ ways; the particular member from each of the latter in 2 ways. The total number of ways as i varies from 0 to $(r-1)/2$ or $r/2$ is the c_r given by (3a)$'$. Similarly the c_r of (3b) can be proved equal to $C_{2j',2j'-r}=C_{2j',r}$. Thus (2) becomes (cf. [5], 6⁰, p. 178)

$$\sum_{r=0}^{2j'} C_{2j',r} C_{n'-2j',k'-r} = C_{n',k'} = C_{n-2,k-1}. \tag{4}$$

By a similar process we prove

$$\sum_{r=0}^{j-1} 2^r \sum_{l=0}^{j-r-1} \frac{(j-1)!}{r!l!(j-r-l-1)!} C_{n-2j,k-r-2l-2} = C_{n-2,k-2}. \tag{5}$$

Hence

$$\begin{aligned} \sum_{r=0}^{j} \sum_{l=0}^{j-r} C_{j,r} C_{j-r,l} C_{n-2j,k-r-2l}(kr-r-2l)2^{r-1} &= j[C_{n-2,k-1}(k-1) - C_{n-2,k-2}] \\ &= j(n-k-1)C_{n-2,k-2}, \end{aligned} \tag{6}$$

which was to be proved.

6. **Special involution curves for $k=2$.** We want to know what situations arise when some or all of the j double roots in the special n-ic [§§2, 3, 4]

coincide, as may occur for the involution curves of W_p. Definite results can easily be obtained for $k=2$. In the special n-ic let $2k_1$, of the roots coincide in t_1, $2k_2$ in $t_2, \cdots, 2k_m$ in t_m, while $n-2\sum_{i=1}^{m}k_i$ of the roots remain simple. If x_0 corresponds to the special n-ic, then as x, very near x_0, turns about x_0, the roots of the n-ic corresponding to x are permuted according to the permutation

$$(1)\quad S = (t_{11}t_{12}\cdots t_{1k_1})(t_{21}t_{22}\cdots t_{2k_2})\cdots(t_{m_1}t_{m_2}\cdots t_{mk_m})(\tau_1)(\tau_2)\cdots(\tau_{n-2\Sigma k_i}).$$

A corresponding permutation Σ takes place among the $C_{n,2}$ pairs of t's and τ's chosen from the roots of the n-ic. As x approaches x_0, a letter t_i chosen from the ith cycle approaches the point P_i of N^2, and τ_h approaches the point Q_h of N^2. Likewise a pair (t_it_i') chosen from the same cycle approaches the point π_i of V_2, a pair (t_it_l) from different cycles approaches the point π_{il} of V_2, a pair $(t_i\tau_h)$ approaches the point ρ_{ih} of V_2, and a pair $(\tau_h\tau_s)$ approaches the point σ_{hs} of V_2.

We first seek what contribution P_i makes to η_s due to permutations among the k-ads (t_it_l). Let d_{il} be the lowest common multiple of $2k_i$, $2k_l$. The permutation S^{2k_i} carries t_i around P_i. The permutation induced by this among the t's of the lth cycle is of period $d_{il}/2k_i$, and so consists of $4k_ik_l/d_{il}$ cycles of $d_{il}/2k_i$ letters each. Hence the pairs (t_it_l) on t_i are permuted likewise in $4k_ik_l/d_{il}$ cycles of $d_{il}/2k_i$ pairs each. The same argument is followed when we consider a point Q_h and pairs containing τ's. Hence

$$(2)\quad \begin{aligned} \eta_s = {} & \sum_{i,l}[4k_ik_l/d_{il}][d_{il}(1/2k_i + 1/2k_l) - 2] \\ & + \left[n - 2\sum_i k_i\right]\left[\sum_i (2k_i - 1)\right], \end{aligned}$$

where i, l take all integral values from 1 to m, and where $i<l$ in each term of the first summation. We next determine η_s'. No points π_{il}, ρ_{ih}, σ_{hs} of V_2 can be branch points, since the corresponding permutations are powers of S which leave fixed the t's and τ's involved in each case. If π_i is a branch point, the point (t_it_i') moving around it must reach its original position by an interchange of t_i, t_i'. The corresponding permutation among the letters of the ith cycle must therefore be of period two and consist of k_i cycles. As (t_it_i') moves around π_i, it passes successively through the points represented by these cycles. Hence each π_i counts once as a branch point and

$$(3)\quad \eta_s' = m.$$

Finally there are in the pencil of n-ics $2(n-1)-\sum_{i=1}^{m}(2k_i-1)$ members with a double root each. Their contribution is

(4) $$\eta_0 - \eta_0' = \left[2(n-1) - \sum_{i=1}^{m} (2k_i - 1)\right][n-3].$$

From (2), (3), and (4), we may easily determine p_2 by Zeuthen's formula.

In concrete cases it is of greater advantage to know at once the effect on V_2's genus of the points π_i, π_{il}, ρ_{ih}, σ_{hs}. Since a σ_{hs} is an ordinary simple point of V_2, and a ρ_{ih} is a simple point with a $2k_i$-point contact tangent τ_h, our attention is concentrated on points π_i and π_{il}. The effect of either type is independent of the roots of the n-ic not appearing in the pair determining the point. Thus π_i is best studied by considering a pencil of $2k_i$-ics which contains one member with a $2k_i$-fold root. By (2), (3), and (4), $\eta_s = 0$, $\eta_s' = 1$, $\eta_0 - \eta_0' = (2k_i-1)(2k_i-3)$. Hence

(5) $$\eta - \eta' = 4k_i^2 - 8k_i + 2,$$

and

(6) $$p_2 = k_i^2 - 3k_i + 2.$$

Since $C_{2k_i-2,2}$ is the maximum value of the genus for curves of C_2's order we have proved the theorem:

THEOREM 1. *The point π_i introduced into V_2, if the pencil contains an n-ic with a $2k_i$-fold root, is equivalent to $(k_i-1)^2$ ordinary double points in the calculation of the genus.*

THEOREM 2. *If the pencil of n-ics contains a member with one $2k_i$-fold root, and another $2k_l$-fold root, the point π_{il} introduced into V_2 decreases the genus by the number $M = 2k_ik_l - k_i - k_l + f(k_i, k_l)$, where $f(k_i, k_l)$ is the greatest common divisor of k_i and k_l. Since M may be written $C_{2k_l,2} + [2k_l(k_i - k_l) - k_i + f(k_i, k_l)]$, the effect of π_{il} on the genus is that of a $2k_l$-fold point into which is absorbed an extra number of double points equal to the number N in the brackets.*

Note the symmetry of N in k_l and $k_i - k_l$. To prove the theorem, we take a pencil of $2(k_i+k_l)$-ics containing one member $(ts_i)^{2k_i}(ts_l)^{2k_l}$. Substituting $2k_ik_l/f(k_i, k_l)$ for d_{il} in (2), and using also (3), (4), we have

(7) $$\eta - \eta' = 4k_i^2 + 8k_ik_l + 4k_l^2 - 4k_i - 4k_l - 4f(k_i, k_l) - 2.$$

Zeuthen's formula gives us

(8) $$p_2 = k_i^2 + 2k_ik_l + k_l^2 - 2k_i - 2k_l - f(k_i, k_l) + 1.$$

The maximum genus for a curve of V_2's order is $C_{2k_i+2k_l-2,2}$. Allowing for the reductions $(k_i-1)^2$, $(k_l-1)^2$ in p_2 due to π_i, π_l respectively, we get $2k_ik_l - k_i - k_l + f(k_i, k_l)$ as the reduction due to π_{il}.

We now construct several curves in order to give concrete examples of the multiple points. Let us first consider the pencil $\lambda_0(tt_1)^8\lambda_1(\alpha t)^8$, where $t_1=0$, $(\alpha t)^8=\alpha_0t^8+\alpha_1t^7+\alpha_2t^6+\alpha_3t^5+\alpha_4t^4+\alpha_5t^3+\alpha_6t^2+\alpha_7t+\alpha_8$. The equation of V_2^7 is

$$\begin{vmatrix} t^8 & (\alpha t)^8 \\ t'^8 & (\alpha t')^8 \end{vmatrix} \cdot \frac{1}{t-t'} = 0. \tag{9}$$

If we put $x_0=tt'$, $x_1=t+t'$, $x_2=1$, the point (t_1t_1) on V_2^7 has coordinates $(0, 0, 1)$; the line t_1 has the equation $x_0=0$; and the equation of V_2^7, arranged in powers of x_2, becomes

$$\begin{aligned} &x_0^3x_2^3(\alpha_7x_0 + 4\alpha_8x_1) - x_0^2x_2^2(\alpha_5x_0^3 + 3\alpha_6x_0^2x_1 + 6\alpha_7x_0x_1^2 + 10\alpha_8x_1^3) \\ &+ x_0x_2(\alpha_3x_0^5 + 2\alpha_4x_0^4x_1 + 3\alpha_5x_0^3x_1^2 + 4\alpha_6x_0^2x_1^3 + 5\alpha_7x_0x_1^4 + 6\alpha_8x_1^5) \\ &- \alpha_1x_0^7 - \alpha_2x_0^6x_1 - \alpha_3x_0^5x_1^2 - \alpha_4x_0^4x_1^3 - \alpha_5x_0^3x_1^4 - \alpha_6x_0^2x_1^5 \\ &- \alpha_7x_0x_1^6 - \alpha_8x_1^7 = 0. \end{aligned} \tag{10}$$

The point (t_1t_1) is shown to be a fourfold point on V_2^7. The line t_1, counting three times as a tangent line, meets V_2^7 in seven points here. If we put $x_0=y$, $x_1=x$, $x_2=1$, and use Newton's method of approximation, we find that in the neighborhood of the origin (t_1t_1), the curve has three branches of type $y=Ax^2$ and one of type $y=Bx$. On a slight dislocation of the configuration every pair of the three former branches intersect in two points, while the fourth intersects the others in one point each. This accounts for the nine double points absorbed at (t_1t_1).

We next consider the pencil $\lambda_0(tt_1)^8(tt_9)^2+\lambda_1(\beta t)^{10}$, where $t_1=0$, $t_9=\infty$. If we again transform to homogeneous coordinates, the point (t_1t_9) has coordinates $(0, 1, 0)$; the equations of t_1, t_9 are $x_0=0$, $x_2=0$ respectively; and the equation of V_2^9, arranged in powers of x_1, becomes

$$\begin{aligned} &\beta_{10}x_1^7x_2^2 + \beta_9x_0x_1^6x_2^2 + x_1^5x_2^2(\beta_8x_0^2 - 6\beta_{10}x_0x_2) + x_1^4x_2^2(\beta_7x_0^3 - 5\beta_9x_0^2x_2) \\ &+ x_1^3x_2^2(\beta_6x_0^4-4\beta_8x_0^3x_2 +10\beta_{10}x_0^2x_2^2) +x_1^2x_2^2(\beta_5x_0^5 -3\beta_7x_0^4x_2 + 6\beta_9x_0^3x_2^2) \\ &- x_1(\beta_0x_0^8 - \beta_4x_0^6x_2^2 + 2\beta_6x_0^5x_2^3 - 3\beta_8x_0^4x_2^4 + 4\beta_{10}x_0^3x_2^5) \\ &- \beta_1x_0^8x_2 + \beta_3x_0^7x_2^2 - \beta_5x_0^6x_2^3 + \beta_7x_0^5x_2^4 - \beta_9x_0^4x_2^5 = 0. \end{aligned} \tag{11}$$

The point (t_1t_9) is a double point; the two tangents coincide in t_9, which meets V_2^9 here in eight points. Putting $x_0=x$, $x_1=1$, $x_2=y$, we find by Newton's approximation that the two branches at the origin (t_1t_9) are of types $y=Ax^4$, $y=-Ax^4$, respectively. Each branch touches the tangent at a point of undulation. If the coefficients of V_2^9's equation were very slightly changed, the two branches would intersect in the four double points which are absorbed at (t_1t_9).

We finally take the pencil $\lambda_0(tt_1)^6(tt_7)^4+\lambda_1(\gamma t)^{10}$, where $t_1=0$, $t_7=\infty$. The point (t_1t_7) has coordinates (0, 1, 0); the equations for t_1, t_7 are $x_0=0$, $x_2=0$ respectively, and V_2^9's equation, arranged in powers of x_1, reads

$$\begin{aligned}&\gamma_{10}x_1^5x_2^4+\gamma_9x_0x_1^4x_2^4-x_0x_1^3(\gamma_0x_0^5-\gamma_8x_0x_2^4+4\gamma_{10}x_2^5)\\&-x_0^2x_1^2x_2(\gamma_1x_0^4-\gamma_7x_0x_2^3+3\gamma_9x_2^4)+x_0^2x_1x_2(2\gamma_0x_0^5-\gamma_2x_0^4x_2\\&+\gamma_6x_0^2x_2^3-2\gamma_8x_0x_2^4+3\gamma_{10}x_2^5)+\gamma_1x_0^7x_2^2-\gamma_3x_0^6x_2^3+\gamma_5x_0^5x_2^4\\&-\gamma_7x_0^4x_2^5+\gamma_9x_0^3x_2^6=0.\end{aligned}\tag{12}$$

The point (t_1t_7) is a fourfold point on V_2^9, at which the four tangents coincide in t_7, meeting V_2^9 in six points here. If we put $x_0=x$, $x_1=1$, $x_2=y$, we find by Newton's method that the curve has at the origin (t_1t_7) two branches of types $y^2=Ax^3$, $y^2=-Ax^3$ respectively. If we allowed these two cusps to change into interlacing loops, we should bring into evidence the eight double points absorbed at (t_1t_7).

For higher values of k the same general method may be employed to determine η_s, η_s', but the application will naturally involve difficulties due to the greater complexities of these cases.

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